

Noncolliding processes, matrix-valued processes and determinantal processes ^{*}

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Abstract

A noncolliding diffusion process is a conditional process of N independent one-dimensional diffusion processes such that the particles never collide with each other. This process realizes an interacting particle system with long-ranged strong repulsive forces acting between any pair of particles. When the individual diffusion process is a one-dimensional Brownian motion, the noncolliding process is equivalent in distribution with the eigenvalue process of an $N \times N$ Hermitian-matrix-valued process, which we call Dyson's model. For any deterministic initial configuration of N particles, distribution of particle positions of the noncolliding Brownian motion on the real line at any fixed time $t > 0$ is a determinantal point process. We can prove that the process is determinantal in the sense that the multi-time correlation function for any chosen series of times, which determines joint distributions at these times, is also represented by a determinant. We study the asymptotic behavior of the system, when the number of Brownian motions N in the system tends to infinity. This problem is concerned with the random matrix theory on the asymptotics of eigenvalue distributions, when the matrix size becomes infinity. In the present paper, we introduce a variety of noncolliding diffusion processes by generalizing the noncolliding Brownian motion, some of which are temporally inhomogeneous. We report the results of our research project to construct and study finite and infinite particle systems with long-ranged strong interactions realized by noncolliding processes.

Key words and phrases. Noncolliding diffusion processes, determinantal (Fermion) point processes, random matrix theory, Fredholm determinants, Tracy-Widom distributions and Painlevé equations, Harish-Chandra (Itzykson-Zuber) integral formulas, infinite particle systems

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1 Introduction

In a system of N independent one-dimensional diffusion processes, if we impose a condition such that the particles never collide with each other, then we obtain an interacting particle system with long-ranged strong repulsive forces acting between any pair of particles. We call such a system a **noncolliding diffusion process**. In 1962 Dyson [14] showed that, when the individual diffusion process is a one-dimensional Brownian motion, the obtained noncolliding process, the **noncolliding Brownian motion**, is related to a **matrix-valued process**. He introduced a Hermitian-matrix-valued process having Brownian motions as its diagonal elements, and complex Brownian motions as off-diagonal elements. The size of the matrix is supposed to be $N \times N$. By virtue of the Hermitian property, all eigenvalues of the matrix are real, and Dyson derived a system of N -simultaneous stochastic differential equations for the process of N eigenvalues on the real line \mathbf{R} . In the present paper we call this stochastic process of eigenvalues **Dyson's model**. (Strictly speaking, it is a special case of Dyson's Brownian motion models with the parameter $\beta = 2$ as explained below.) If we regard each eigenvalue as a particle position in one dimension, Dyson's model is considered to be a one-dimensional system of interacting Brownian motions. Dyson showed that this system is nothing but the noncolliding Brownian motion [4, 23].

A probability distribution on the space of particle configurations is called a **determinantal point process** or a **Fermion point process**, if its correlation functions are generally represented by determinants [65, 66, 27]. The noncolliding Brownian motion provides us examples of determinantal point processes: for any deterministic initial configuration of N particles, distribution of particle positions on \mathbf{R} at any fixed time $t > 0$ is a determinantal point process [45]. Moreover, by using the method developed by Eynard and Mehta for multi-layer random matrix models [16, 50], we can show that the multi-time correlation functions for any chosen series of times, which determine joint distributions at these times, are also represented by determinants [37, 43, 44, 45]. In the present paper we call such a stochastic process that any multi-time correlation function is given by a determinant a **determinantal process** [43].

We study the asymptotic behavior of the system, when the number of Brownian motions N in the system tends to infinity. Since, as explained above, the noncolliding Brownian motion can be realized by the eigenvalue process and the correlation functions are expressed by determinants of matrices, this problem is concerned with the asymptotics of eigenvalue distributions, when the matrix size becomes infinity. The latter problem is one of the main topics of the random matrix theory [50]. In other words, our research project reported in this paper is to construct infinite particle systems with long-ranged strong interactions by applying the results of recent development of the random matrix theory [37, 40, 42, 43, 44, 45].

In the present paper, we introduce a variety of noncolliding diffusion processes by generalizing the noncolliding Brownian motion. In Section 2 first we explain basic

properties of diffusion processes treated in this paper, such as Brownian motions, Brownian bridges, absorbing Brownian motions, Bessel processes, Bessel bridges, and generalized meanders. The transition probability density of a noncolliding diffusion process is expressed by a determinant of a matrix, each element of which is the transition probability density of the individual diffusion process in one dimension (the Karlin-McGregor formula). This formula provides a useful tool for us to analyze noncolliding diffusion processes. In Section 3 we state the Karlin-McGregor formula and present basic properties of noncolliding diffusion processes. When such a Hermitian-matrix-valued process is given that its elements are one-dimensional diffusion processes, it will be a fundamental and interesting problem to determine a system of stochastic differential equations for eigenvalue process of the given matrix-valued process. Bru's theorem [8, 9] and its generalization [39, 40] give answers to this problem. In Section 4 we give a generalized version of Bru's theorem and show its applications. The determinantal structures of multi-time correlation functions of noncolliding processes are explained in Section 5. Their asymptotics in $N \rightarrow \infty$ are also discussed [37, 42, 43, 44, 45].

When we impose the noncolliding condition on a finite time-interval $(0, T)$, $T \in (0, \infty)$ instead of an infinite time-interval $(0, \infty)$, the noncolliding diffusion processes become temporally inhomogeneous, even if individual one-dimensional diffusion processes are temporally homogeneous. In Section 6 we discuss these temporally inhomogeneous noncolliding processes. These processes are not determinantal any more, and make a new family of processes, which we call **Pfaffian processes** [37, 42]. In the last section, Section 7, we list up the topics, which are related to noncolliding processes, but can not be discussed here.

2 Brownian motion and its conditional processes

Let (Ω, \mathcal{F}, P) be a probability space. The stochastic process called (one-dimensional or linear) **Brownian motion**, $\{B(t, \omega)\}_{t \in [0, \infty)}$, satisfies the following conditions :

1. $B(0, \omega) = 0$ with probability one.
2. For any fixed $\omega \in \Omega$, $B(t, \omega)$ is a real continuous function of t . (This property is expressed by saying that $B(t)$ has a **continuous path**.)
3. For any sequence of times, $t_0 \equiv 0 < t_1 < \dots < t_M$, $M = 1, 2, \dots$, the increments $\{B(t_i) - B(t_{i-1})\}_{i=1,2,\dots,M}$ are independent, and distribution of each increment is normal with mean zero and variance $t_i - t_{i-1}$.

Then, the probability that the Brownian motion is observed in the interval $[a_i, b_i] \subset \mathbf{R}$ at time t_i for each $i = 1, 2, \dots, M$, $P(B(t_i) \in [a_i, b_i], i = 1, 2, \dots, M)$, is given by

$$\int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \cdots \int_{a_M}^{b_M} dx_M \prod_{i=1}^M G(t_i - t_{i-1}, x_i - x_{i-1}),$$

where $x_0 \equiv 0$ and

$$G(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad t > 0, \quad x \in \mathbf{R}.$$

The integral kernel $G(s, x; t, y) \equiv G(t-s, y-x)$ is called the **transition probability density function** of the Brownian motion. For any fixed $s \geq 0$, under the condition that $B(s)$ is given, $B(u), u \leq s$ and $B(t), t > s$ are independent. This property is called a **Markov property**. A positive random variable τ is called a **Markov time**, if the event $\{\tau \leq u\}$ is determined by the behavior of the process until time u and independent of the behavior of the process after time u . The first time that a Brownian motion visits a given domain D , which is called the hitting time of D , is an example of a Markov time. In the definition of Markov property mentioned above, if a deterministic time s is replaced by a Markov time τ , then we obtain a stronger property, called a **strong Markov property**. In general, a stochastic process, which has a strong Markov property and has a continuous path almost surely, is called a diffusion process. See [73, 62] for instance. In the case that the transition probability density function $G(s, x; t, y)$ does not depend on times t and s themselves but only depends on the difference $t-s$, a Markov process is said to be **temporally homogeneous**. In this case we write the transition probability density function as $G(t-s, y|x)$ instead of $G(s, x; t, y)$ to clarify its homogeneity in time in this paper. The Brownian motion is an example of a temporally homogeneous diffusion process. (It is also spatially homogeneous.)

For $d \in \mathbf{N} \equiv \{1, 2, \dots\}$, using independent one-dimensional Brownian motions $B_1(t), B_2(t), \dots, B_d(t), t \geq 0$, a d -dimensional Brownian motion is defined by a vector-valued diffusion process $\mathbf{B}(t) = (B_1(t), B_2(t), \dots, B_d(t)), t \geq 0$.

We want to consider the Brownian motion under the condition that it visits the origin at a given time $T > 0$. Since the probability that this condition is satisfied is zero, we first consider the Brownian motion under another condition such that it visits some point in an interval $(-\varepsilon, \varepsilon)$ at time $T, \varepsilon > 0$, and then define the original conditional process by taking the limit $\varepsilon \downarrow 0$. The transition probability density function obtained in this limit is

$$G^T(s, x; t, y) = \frac{G(T-t, 0|y)G(t-s, y|x)}{G(T-s, 0|x)}, \quad 0 \leq s < t \leq T, \quad x, y \in \mathbf{R}.$$

It is a temporally inhomogeneous diffusion process. We call this process a **Brownian bridge** of duration T and denote it by $\beta^T(t), t \in [0, T]$.

Although one-dimensional Brownian motion can visit any point of \mathbf{R} , we consider the Brownian motion conditioned to stay positive forever. This conditional process $Y(t), t \in [0, \infty)$ is temporally homogeneous process with transition probability density function $G^{(1/2)}(t, y|x)$;

$$\begin{aligned} G^{(1/2)}(t, y|x) &= \frac{y}{x} \left\{ G(t, y|x) - G(t, -y|x) \right\}, \quad t > 0, \quad x > 0, \quad y \geq 0, \quad (2.1) \\ G^{(1/2)}(t, y|0) &= \frac{2}{t} y^2 G(t, y|0), \quad t > 0, \quad y \geq 0. \end{aligned}$$

The distance of a three-dimensional Brownian motion from the origin, $(B_1(t)^2 + B_2(t)^2 + B_3(t)^2)^{1/2}, t > 0$ has exactly the same transition probability density as (2.1), and is called the **three-dimensional Bessel process**. In other words, the three-dimensional Bessel process $Y(t), t \in [0, \infty)$ has two different representations, ‘the representation by a Brownian motion conditioned to stay positive’ and ‘the representation by a radial part of the three-dimensional Brownian motion’. We also note that $Y(t)$ solves the following stochastic differential equation [62],

$$Y(t) = B(t) + \int_0^t \frac{1}{Y(s)} ds, \quad t > 0.$$

Consider the Brownian motion $X(t), t \in [0, T]$ under the condition that it stays positive during a finite time-interval $(0, T]$, with $T \in (0, \infty)$. Then the conditional process is a temporally inhomogeneous diffusion process with the transition probability density function $G_T^{(1/2,1)}(s, x; t, y)$;

$$G_T^{(1/2,1)}(s, x; t, y) = \frac{h(T-t, y)}{h(T-s, x)} \left\{ G(t-s, y|x) - G(t-s, -y|x) \right\}, \quad (2.2)$$

$$0 \leq s < t \leq T, \quad x > 0, \quad y \geq 0,$$

$$G_T^{(1/2,1)}(0, 0; t, y) = \frac{\sqrt{2\pi T}}{t} h(T-t, y) y G(t, y|0), \quad t \in (0, T], \quad y \geq 0,$$

where $h(s, x), x > 0, s > 0$ is the probability that the Brownian motion starting from $x > 0$ stays positive during the time-interval $[0, s]$. This conditional process is called a **Brownian meander**. Using two independent Brownian motions $B_1(t), B_2(t)$ and a Brownian bridge $\beta^T(t)$ of duration T , which is independent of $B_1(t)$ and $B_2(t)$, we define a one-dimensional diffusion process by $(B_1(t)^2 + B_2(t)^2 + \beta^T(t)^2)^{1/2}, t \in [0, T]$. We can prove that this process is identified with the Brownian meander. That is, the Brownian meander has also two different representations, ‘the representation by a Brownian motion conditioned to stay positive during a finite time-interval $(0, T]$ ’ and ‘the representation by a radial part of the three-dimensional diffusion process $(B_1(t), B_2(t), \beta^T(t)), t \in (0, T]$ ’ [74].

By comparing (2.1) with (2.2), we see that the distributions of the three-dimensional Bessel process $Y(t)$ and the Brownian meander $X(t)$, both starting from the origin, are absolutely continuous and satisfy

$$P(X(\cdot) \in dw) = \sqrt{\frac{\pi T}{2}} \frac{1}{w(T)} P(Y(\cdot) \in dw). \quad (2.3)$$

The equality (2.3) is called **Imhof’s relation** [28].

The Brownian motion, which is killed at the origin, is called an **absorbing Brownian motion** in the domain $(0, \infty)$. Let $\hat{G}(t-s, y|x)$ be the transition probability density of this process. It is the density of the Brownian motion at time t , which starts from $x > 0$ at time $s (< t)$, restricted on the event that it stays positive

in the time-interval $[s, t]$. The **reflection principle** of Brownian motion gives

$$\widehat{G}(t-s, y|x) = G(t-s, y|x) - G(t-s, -y|x).$$

The first formula in (2.1) means that the transformation of the transition probability density $\widehat{G}(t, y|x)$, given by $(y/x)\widehat{G}(t, y|x)$, is identified with the transition probability density $G^{(1/2)}(t, y|x)$ of the three-dimensional Bessel process. It implies that the three-dimensional Bessel process $Y(t)$ is the Doob ***h*-transformation** of the absorbing Brownian motion in the domain $(0, \infty)$.

When $d \in \mathbf{N}$, the distance of the d -dimensional Brownian motion from the origin, $(B_1(t)^2 + B_2(t)^2 + \cdots + B_d(t)^2)^{1/2}$, defines a one-dimensional diffusion process, which we call the ***d*-dimensional Bessel process**. The Bessel process can be extended to the cases with all positive real values of d as follows. With a parameter $\nu \in (-1, \infty)$, the transition probability density function of the $2(\nu+1)$ -dimensional Bessel process $Y^{(\nu)}(t)$, is given by

$$\begin{aligned} G^{(\nu)}(t, y|x) &= \frac{y^{\nu+1}}{x^\nu} \frac{1}{t} \exp\left(-\frac{x^2 + y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right), \quad t > 0, \quad x > 0, \quad y \geq 0, \\ G^{(\nu)}(t, y|0) &= \frac{y^{2\nu+1}}{2^\nu \Gamma(\nu+1) t^{\nu+1}} \exp\left(-\frac{y^2}{2t}\right), \quad t > 0, \quad y \geq 0, \end{aligned}$$

where $\Gamma(z)$ is the Gamma function and $I_\nu(z)$ is the modified Bessel function with parameter ν [62]. The behavior of the Bessel process depends on the dimension d (the parameter $\nu = (d-2)/2$). When d is greater than or equal to 2 ($\nu \geq 0$), the process has the origin as a transient point, and when d is less than 2 ($-1 < \nu < 0$), it has the origin as a recurrent point. Moreover, if and only if d is greater than or equal to 1 ($\nu \geq -1/2$), it is a semi-martingale [62].

Yor [74] introduced a family of diffusion processes with two parameters (ν, κ) , $\nu \in (-1, \infty)$, $\kappa \in (0, 2(\nu+1))$, which includes the Brownian meander as a special case $(\nu, \kappa) = (1/2, 1)$, and he called each member of the family a **generalized meander**. The generalized meander $X^{(\nu, \kappa)}(t)$, $\nu \in (-1, \infty)$, $\kappa \in (0, 2(\nu+1))$, is the diffusion process with the transition probability density

$$\begin{aligned} G_T^{(\nu, \kappa)}(s, x; t, y) &= \frac{h_T^{(\nu, \kappa)}(t, y)}{h_T^{(\nu, \kappa)}(s, x)} G^{(\nu)}(t-s, y|x), \quad 0 \leq s < t \leq T, \quad x > 0, \quad y \geq 0, \\ G_T^{(\nu, \kappa)}(0, 0; t, y) &= \frac{\Gamma(\nu+1)(2T)^{\kappa/2}}{\Gamma(\nu+1-\kappa/2)} h_T^{(\nu, \kappa)}(t, y) G^{(\nu)}(t, y|0), \quad t \in (0, T], \quad y \geq 0, \end{aligned}$$

where $h_T^{(\nu, \kappa)}(t, x) = \int_0^\infty dy G^{(\nu)}(T-t, y|x) y^{-\kappa}$, $x \geq 0$, $t \in (0, T]$ [74]. Then Imhof's relation (2.3) between the three-dimensional Bessel process $Y(t) = Y^{(1/2)}(t)$ and the Brownian meander $X(t) = X^{(1/2, 1)}(t)$ is generalized as

$$P(X^{(\nu, \kappa)}(\cdot) \in dw) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\kappa/2)} \left(\frac{\sqrt{2T}}{w(T)} \right)^\kappa P(Y^{(\nu)}(\cdot) \in dw)$$

for the $2(\nu + 1)$ -dimensional Bessel process $Y^{(\nu)}(t)$ and the generalized meander $X^{(\nu, \kappa)}(t)$. We remark that, though the parameter κ of generalized meanders is in $(0, 2(\nu + 1))$, we can discuss the cases $\kappa = 0$ and $\kappa = 2(\nu + 1)$. The former corresponds to the Bessel processes, and the latter the Bessel bridges, which are the conditional Bessel processes to arrive at the origin at a fixed time $T > 0$.

3 Noncolliding diffusion processes

3.1 Karlin-McGregor formula

In order to analyze noncolliding diffusion processes, it is useful to represent the transition probability density functions by means of determinants. The representation is called the **Karlin-McGregor formula** [35] in probability theory, and the **Lindström-Gessel-Viennot formula** [48, 21, 68] in combinatorics. It is also regarded as a stochastic-process version of the **Slater determinant**, which originally expresses a many-body wave function of free Fermion particles in quantum mechanics [69, 43].

[Karlin-McGregor formula] ([35, 48, 21]) Let $G(s, x; t, y)$ be the transition probability density function of a one-dimensional diffusion process. On the line \mathbf{R} set N starting points x_i , $i = 1, 2, \dots, N$ and N terminal points y_i , $i = 1, 2, \dots, N$ with $x_1 < x_2 < \dots < x_N$ and $y_1 < y_2 < \dots < y_N$, respectively. The transition probability density function of the system of N diffusion processes restricted on the event that they never collide with each other during the time-interval $[s, t]$ is given by

$$G_0(s, \mathbf{x}; t, \mathbf{y}) = \det_{1 \leq i, j \leq N} \left(G(s, x_i; t, y_j) \right).$$

When $N = 2$, this claim is essentially equivalent to the reflection principle for a Brownian motion. That is, this formula can be regarded as a generalization of the reflection principle [41].

3.2 Noncolliding Brownian motions in a finite and an infinite time-intervals

Let \mathbf{W}_N^A be a subset of \mathbf{R}^N defined by $\mathbf{W}_N^A = \{\mathbf{x} \in \mathbf{R}^N : x_1 < x_2 < \dots < x_N\}$, which is called the Weyl chamber of type A_{N-1} in representation theory [20]. The transition probability density of the absorbing Brownian motion in \mathbf{W}_N^A , that is, the density function of an N -dimensional Brownian motion at time t , which starts from $\mathbf{x} \in \mathbf{W}_N^A$ at time 0, restricted on the event that it stays in \mathbf{W}_N^A during the time-interval $[0, t]$, is represented by

$$f_N(t, \mathbf{y} | \mathbf{x}) = \det_{1 \leq i, j \leq N} \left(G(t, y_j | x_i) \right)$$

by the Karlin-McGregor formula. Then the probability that the Brownian motion stays in \mathbf{W}_N^A until time t is

$$\mathcal{N}_N(t, \mathbf{x}) = \int_{\mathbf{W}_N^A} f_N(t, \mathbf{y}|\mathbf{x}) d\mathbf{y}.$$

Now we consider the **noncolliding Brownian motion in a finite time-interval** $t \in (0, T]$, $\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_N(t))$. The transition probability density of the process denoted by $g_{N,T}(s, \mathbf{x}; t, \mathbf{y})$ is the conditional density of N Brownian motions at time t , which started from the points $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbf{W}_N^A$ at time $s (< t)$, under the condition that they never collide with each other in the time-interval $[s, t]$. It is given by

$$g_{N,T}(s, \mathbf{x}; t, \mathbf{y}) = \frac{\mathcal{N}_N(T-t, \mathbf{y})}{\mathcal{N}_N(T-s, \mathbf{x})} f_N(t-s, \mathbf{y}|\mathbf{x}), \quad 0 \leq s < t \leq T, \quad \mathbf{x}, \mathbf{y} \in \mathbf{W}_N^A. \quad (3.1)$$

The transition probability density in the case that all N particles start from the origin will be defined by taking the limit $\mathbf{x} \rightarrow \mathbf{0} \equiv (0, 0, \dots, 0)$ in (3.1). Since both of the numerator and the denominator in (3.1) tend to 0 as $\mathbf{x} \rightarrow \mathbf{0}$, we have to know the asymptotic behavior of $f_N(t, \mathbf{y}|\mathbf{x})$ and $\mathcal{N}_N(t, \mathbf{x})$ in $|\mathbf{x}|/\sqrt{t} \rightarrow 0$. Performing bilinear expansions [38, 40, 43] with respect to the multivariate symmetric functions called the **Schur functions** [20, 19], we have obtained

$$\begin{aligned} f_N(t, \mathbf{y}|\mathbf{x}) &\sim \frac{t^{-N(N+1)/4}}{C_1(N)} h_N\left(\frac{\mathbf{x}}{\sqrt{t}}\right) h_N(\mathbf{y}) \exp\left(-\frac{|\mathbf{y}|^2}{2t}\right), \\ \mathcal{N}_N(t, \mathbf{x}) &\sim \frac{C_2(N)}{C_1(N)} h_N\left(\frac{\mathbf{x}}{\sqrt{t}}\right), \quad \frac{|\mathbf{x}|}{\sqrt{t}} \rightarrow 0. \end{aligned} \quad (3.2)$$

Here $h_N(\mathbf{x})$ is the $N \times N$ **Vandermonde determinant**, which is equal to the product of differences of variables x_1, x_2, \dots, x_N ,

$$h_N(\mathbf{x}) = \det_{1 \leq i, j \leq N} (x_j^{i-1}) = \prod_{1 \leq i < j \leq N} (x_j - x_i),$$

and $C_1(N) = (2\pi)^{N/2} \prod_{i=1}^N \Gamma(i)$, $C_2(N) = 2^{N/2} \prod_{i=1}^N \Gamma(i/2)$. By using (3.2) we obtain the transition probability density function of the noncolliding Brownian motion, when all N particles start from the origin (*i.e.* $\mathbf{X}(0) = \mathbf{0}$) as

$$g_{N,T}(0, \mathbf{0}; t, \mathbf{y}) = \frac{T^{N(N-1)/4} t^{-N^2/2}}{C_2(N)} \mathcal{N}_N(T-t, \mathbf{y}) h_N(\mathbf{y}) \exp\left(-\frac{|\mathbf{y}|^2}{2t}\right), \quad (3.3)$$

$$t \in (0, T], \quad \mathbf{y} \in \mathbf{W}_N^A.$$

As illustrated in the left picture of Fig.1, in this case the N Brownian motions starting from the origin at time $t = 0$ rapidly separate from each other to avoid collision.

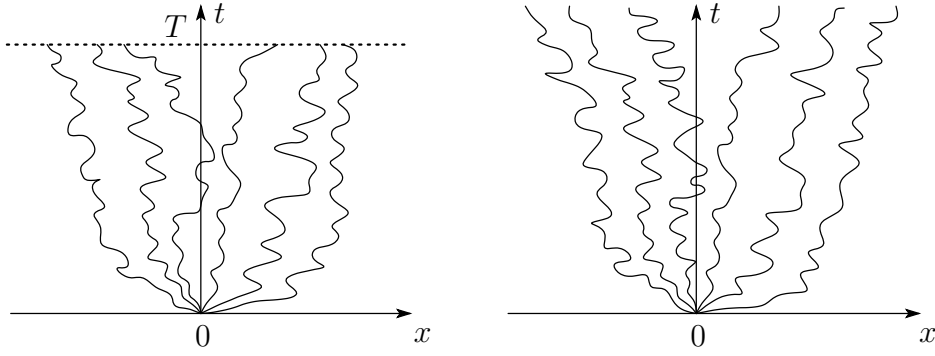


Figure 1: Illustrations of the noncolliding Brownian motions $\mathbf{X}(t), t \in (0, T]$ in the left picture, and $\mathbf{Y}(t), t \in (0, \infty)$ in the right picture, both start from $\mathbf{0}$

When the time T becomes infinity, the process $\mathbf{X}(t)$ converges to the temporally homogeneous process $\mathbf{Y}(t)$, the **noncolliding Brownian motion in an infinite time-interval** $t \in (0, \infty)$, whose transition probability density function $p_N(t, \mathbf{y}|\mathbf{x})$ is given by follows;

$$\begin{aligned} p_N(t, \mathbf{y}|\mathbf{x}) &= \frac{h_N(\mathbf{y})}{h_N(\mathbf{x})} f_N(t, \mathbf{y}|\mathbf{x}), \quad t > 0, \quad \mathbf{x}, \mathbf{y} \in \mathbf{W}_N^A, \\ p_N(t, \mathbf{y}|\mathbf{0}) &= \frac{t^{-N^2/2}}{C_1(N)} h_N(\mathbf{y})^2 \exp\left(-\frac{|\mathbf{y}|^2}{2t}\right), \quad t > 0, \quad \mathbf{y} \in \mathbf{W}_N^A. \end{aligned} \quad (3.4)$$

The above formulas are derived from (3.1) and (3.3) by taking the limit $T \rightarrow \infty$ using (3.2). See the right picture of Fig. 1, which illustrates $\mathbf{Y}(t)$ starting from $\mathbf{0}$.

The first formula of (3.4) implies that $\mathbf{Y}(t)$ is the Doob h -transformation of the absorbing Brownian motion in \mathbf{W}_N^A , whose transition probability density is given by $f_N(t, \mathbf{y}|\mathbf{x})$ [23]. One-parameter family of interacting Brownian motions on \mathbf{R} satisfying the following system of stochastic differential equations

$$Y_i(t) = B_i(t) + \frac{\beta}{2} \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \int_0^t \frac{1}{Y_i(s) - Y_j(s)} ds, \quad 1 \leq i \leq N \quad (3.5)$$

is called **Dyson's Brownian motion model** with parameter $\beta > 0$ [14, 50]. It is readily seen from (3.4) that $\mathbf{Y}(t), t \in (0, \infty)$ solves the system of equations (3.5) with $\beta = 2$.

By comparing (2.2) with (3.1), (3.3), we find that $\mathbf{X}(t)$ can be regarded as a multi-dimensional extension of the Brownian meander $X(t)$. Similarly, by comparing (2.1) with (3.4), $\mathbf{Y}(t)$ can be considered to be a multi-dimensional version of the three-dimensional Bessel process $Y(t)$. Moreover, a multi-dimensional extension of Imhof's relation (2.3) is derived from (3.1), (3.3) and (3.4) as [38]

$$P(\mathbf{X}(\cdot) \in dw) = \frac{C_1(N)}{C_2(N)} \frac{T^{N(N-1)/4}}{h(w(T))} P(\mathbf{Y}(\cdot) \in dw). \quad (3.6)$$

3.3 Noncolliding generalized meander and noncolliding Bessel process

We introduce the subsets \mathbf{W}_N^C and \mathbf{W}_N^D of \mathbf{R}^N defined by

$$\begin{aligned}\mathbf{W}_N^C &= \{\mathbf{x} \in \mathbf{R}^N : 0 < x_1 < x_2 < \cdots < x_N\}, \\ \mathbf{W}_N^D &= \{\mathbf{x} \in \mathbf{R}^N : 0 \leq |x_1| < x_2 < \cdots < x_N\},\end{aligned}$$

which are called the Wyle chambers of type C_N and of type D_N , respectively. The Bessel process $Y^{(\nu)}(t)$, $t \geq 0$ has the origin as a transient point when $\nu \geq 0$, and it has the origin as a recurrent point when $-1 < \nu < 0$. Then the state spaces \mathbf{W}_N of the noncolliding generalized meander and the noncolliding Bessel process, which will be introduced in this subsection, are \mathbf{W}_N^C when $\nu \geq 0$, and \mathbf{W}_N^D when $-1 < \nu < 0$. If the Bessel process is defined by the square root of the squared Bessel process, it can be a multi-valued stochastic process. Consider the squared Bessel process starting from a positive initial point. When $\nu \geq 0$, the process stays positive with probability one, and its square root is determined uniquely, which coincides with the Bessel process introduced in Section 2. While, when $-1 < \nu < 0$, it hits the origin with probability one and then the square root process becomes a bi-valued process after hitting. The generalized meander and the leftmost particles in the N particle systems of the noncolliding generalized meander and of the noncolliding Bessel process are in the same situation. The absolute value $|x_1|$ appearing in the definition of \mathbf{W}_N^D implies that bi-valued processes are allowed, when $-1 < \nu < 0$. See Fig 2. However, we usually consider only nonnegative parts of such bi-valued processes just for simplicity of explanation.

The density function of an N -component generalized meander at time t , which starts from \mathbf{x} in \mathbf{W}_N at time s and stays in \mathbf{W}_N up to time t , is given by

$$f_N^{(\nu, \kappa)}(s, \mathbf{x}; t, \mathbf{y}) = \det_{1 \leq i, j \leq N} \left(G^{(\nu, \kappa)}(s, x_i; t, y_j) \right)$$

from the Karlin-McGregor formula. The probability that the process stays in \mathbf{W}_N during the time-interval $(0, t]$ is given by

$$\mathcal{N}_N^{(\nu, \kappa)}(t, \mathbf{x}) = \int_{\mathbf{W}_N} d\mathbf{y} f_N^{(\nu, \kappa)}(0, \mathbf{x}; t, \mathbf{y}).$$

Then the transition probability density function of the **noncolliding generalized meander** $\mathbf{X}^{(\nu, \kappa)}(t) = (X_1^{(\nu, \kappa)}(t), X_2^{(\nu, \kappa)}(t), \dots, X_N^{(\nu, \kappa)}(t))$ is given by

$$g_{N, T}^{(\nu, \kappa)}(s, \mathbf{x}, t, \mathbf{y}) = \frac{\mathcal{N}_N^{(\nu, \kappa)}(T - t, \mathbf{y})}{\mathcal{N}_N^{(\nu, \kappa)}(T - s, \mathbf{x})} f_N^{(\nu, \kappa)}(t - s, \mathbf{x}, \mathbf{y}), \quad 0 \leq s < t \leq T, \mathbf{x}, \mathbf{y} \in \mathbf{W}_N.$$

Consider the noncolliding generalized meander, when all N particles start from the origin. Define $f_N^{(\nu)}(t, \mathbf{y}|\mathbf{x}) \equiv f_N^{(\nu, 0)}(0, \mathbf{x}; t, \mathbf{y})$. Then we see

$$f_N^{(\nu)}(t, \mathbf{y}|\mathbf{x}) = t^{-N} \prod_{i=1}^N \left(\frac{y_i^{\nu+1}}{x_i^\nu} \right) \exp \left\{ -\frac{1}{2t} \sum_{i=1}^N (x_i^2 + y_i^2) \right\} \det_{1 \leq i, j \leq N} \left(I_\nu \left(\frac{x_i y_j}{t} \right) \right)$$

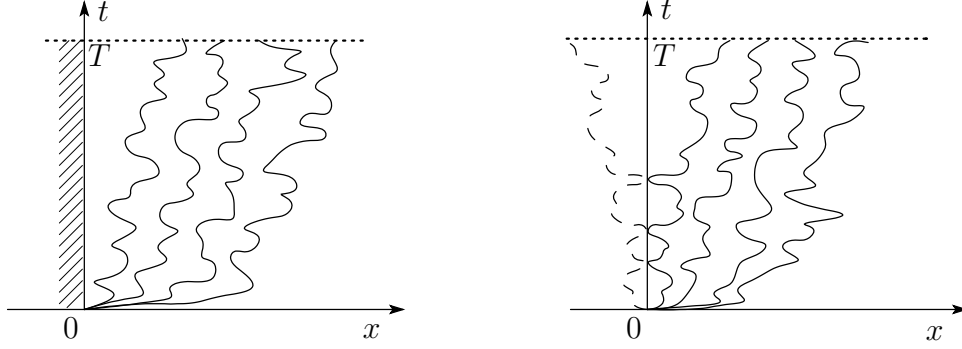


Figure 2: Illustrations of the noncolliding generalized meanders $\mathbf{X}^{(\nu, \kappa)}(t)$ with $\nu \geq 0$ in the left picture, and $-1 < \nu < 0$ in the right picture. When $-1 < \nu < 0$, a bi-valued process is assigned to describe the motion of the leftmost particle.

and using it $g_{N,T}^{(\nu, \kappa)}(s, \mathbf{x}; t, \mathbf{y})$ is rewritten as

$$g_{N,T}^{(\nu, \kappa)}(s, \mathbf{x}; t, \mathbf{y}) = \frac{\tilde{\mathcal{N}}_N^{(\nu, \kappa)}(T - t, \mathbf{y})}{\tilde{\mathcal{N}}_N^{(\nu, \kappa)}(T - s, \mathbf{x})} f^{(\nu)}(t - s, \mathbf{y} | \mathbf{x}), \quad (3.7)$$

where $\tilde{\mathcal{N}}_N^{(\nu, \kappa)}(t, \mathbf{x}) = \int_{\mathbf{W}_N} d\mathbf{y} f_N^{(\nu)}(t, \mathbf{y} | \mathbf{x}) \prod_{i=1}^N y_i^{-\kappa}$. Again performing bilinear expansion with respect to the Schur functions, we have obtained the following asymptotics [40]:

$$\begin{aligned} f_N^{(\nu)}(t, \mathbf{y} | \mathbf{x}) &\sim \frac{t^{-N(N+1+2\nu)/2}}{C^{(\nu)}(N)} h_N^{(0)}\left(\frac{\mathbf{x}}{\sqrt{t}}\right) h_N^{(2\nu+1)}(\mathbf{y}) \exp\left(-\frac{|\mathbf{y}|^2}{2t}\right), \\ \tilde{\mathcal{N}}_N^{(\nu, \kappa)}(t, \mathbf{x}) &\sim \frac{t^{-N\kappa/2} C_N^{(\nu, \kappa)}}{C^{(\nu)}(N)} h_N^{(0)}\left(\frac{\mathbf{x}}{\sqrt{t}}\right), \quad \frac{|\mathbf{x}|}{\sqrt{t}} \rightarrow 0, \end{aligned}$$

where $C^{(\nu)}(N) = 2^{N(N+\nu-1)} \prod_{i=1}^N \Gamma(i) \Gamma(i + \nu)$, $C^{(\nu, \kappa)}(N) = 2^{N(N+2\nu-\kappa-1)/2} \pi^{-N/2} \prod_{i=1}^N \{\Gamma(i/2) \Gamma((i + 2\nu + 1 - \kappa)/2)\}$, and

$$h_N^{(\alpha)}(\mathbf{a}) = \prod_{1 \leq i < j \leq N} (a_j^2 - a_i^2) \prod_{k=1}^N a_k^\alpha.$$

By the above estimates the transition probability density function of the noncolliding generalized meander, when all N particles start from the origin is determined as

$$\begin{aligned} g_{N,T}^{(\nu, \kappa)}(0, \mathbf{0}; t, \mathbf{y}) &= \frac{T^{N(N+\kappa-1)/2} t^{-N(N+\nu)}}{C^{(\nu, \kappa)}(N)} \tilde{\mathcal{N}}_N^{(\nu, \kappa)}(T - t, \mathbf{y}) h_N^{(2\nu+1)}(\mathbf{y}) e^{-|\mathbf{y}|^2/2t} \\ &\quad t \in (0, T], \quad \mathbf{y} \in \mathbf{W}_N. \end{aligned} \quad (3.8)$$

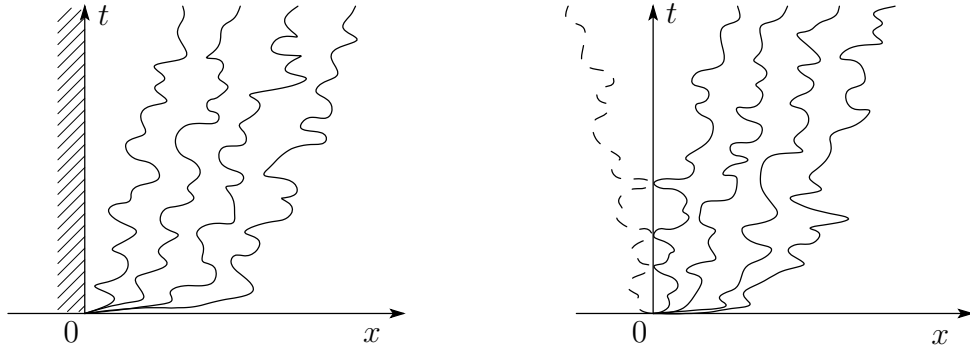


Figure 3: Illustrations of the noncolliding Bessel processes $\mathbf{Y}^{(\nu)}(t)$ with $\nu \geq 0$ in the left picture, and $-1 < \nu < 0$ in the right picture. When $-1 < \nu < 0$, a bi-valued process is assigned to describe the motion of the leftmost particle.

When the time T becomes infinity, the process $\mathbf{X}^{(\nu, \kappa)}(t)$ converges to the temporally homogeneous process $\mathbf{Y}^{(\nu)}(t)$, whose transition probability density function is given by

$$\begin{aligned} p_N^{(\nu)}(t, \mathbf{y} | \mathbf{x}) &= \frac{h_N^{(0)}(\mathbf{y})}{h_N^{(0)}(\mathbf{x})} f_N^{(\nu)}(t, \mathbf{y} | \mathbf{x}), \quad t > 0, \quad \mathbf{x}, \mathbf{y} \in \mathbf{W}_N, \\ p_N^{(\nu)}(t, \mathbf{y} | \mathbf{0}) &= \frac{t^{-N(N+\nu)}}{C^{(\nu)}(N)} h_N^{(\nu+1/2)}(\mathbf{y})^2 \exp\left(-\frac{|\mathbf{y}|^2}{2t}\right), \quad t > 0, \quad \mathbf{y} \in \mathbf{W}_N. \end{aligned} \quad (3.9)$$

See Fig 3. Since the parameter κ controls the distribution of the process when $t \rightarrow T$, it is irrelevant for the process $\mathbf{Y}^{(\nu)}(t)$ in which $T \rightarrow \infty$. The process $\mathbf{Y}^{(\nu)}(t)$ is the **noncolliding** $2(\nu + 1)$ -**dimensional Bessel process**, which is temporally homogeneous and solves the following system of stochastic differential equations when $\nu \geq -1/2$

$$\begin{aligned} Y_i^{(\nu)}(t) &= B_i(t) + \int_0^t \frac{\nu + 1/2}{Y_i(s)} ds + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \int_0^t \frac{2Y_i^{(\nu)}(s)}{Y_i^{(\nu)}(s)^2 - Y_j^{(\nu)}(s)^2} ds, \\ t &\in (0, \infty), \quad 1 \leq i \leq N, \end{aligned} \quad (3.10)$$

where we impose a reflecting wall at the origin when $\nu = -1/2$. Although solution of the system of equations is not necessarily unique in general, $\mathbf{Y}^{(\nu)}(t)$ can be defined as a unique solution such that all coordinates are positive [49]. By comparing (3.7), (3.8) with (3.9), we have the following equality

$$P(\mathbf{X}^{(\nu, \kappa)}(\cdot) \in dw) = \frac{C^{(\nu)}(N)}{C^{(\nu, \kappa)}(N)} \frac{T^{N(N+\kappa-1)/2}}{h_N^{(\kappa)}(w(T))} P(\mathbf{Y}^{(\nu)}(\cdot) \in dw),$$

which is an extension of Imhof's relation for the noncolliding Bessel process $\mathbf{Y}^{(\nu)}(t)$ and the noncolliding generalized meander $\mathbf{X}^{(\nu, \kappa)}(t)$ [40].

4 Matrix-valued processes

4.1 Generalized Bru's theorem

We denote the space of $N \times N$ Hermitian matrices by $\mathcal{H}(N)$ and the spaces of $N \times N$ real symmetric matrices by $\mathcal{S}(N)$. For a matrix A we indicate by tA the transposed matrix of A , by \overline{A} the complex conjugate of A , and by $A^* \equiv {}^t\overline{A}$ the adjoint matrix of A , respectively. We denote the unit matrix of size $N \times N$ by I_N . Bru [8, 9] studied the eigenvalue processes of Wishart process, which is an $\mathcal{H}(N)$ -valued process, and derived the stochastic differential equations for the eigenvalue processes. The result is generalized to the case that each element of matrix-valued process, $\xi_{ij}(t)$, $1 \leq i, j \leq N$, is a complex-valued continuous semi-martingale [39, 40]. In this section we state this generalized version of Bru's theorem and give its applications.

Let $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t))$ be the vector, whose coordinates are eigenvalues of $\mathcal{H}(N)$ -valued process, $\Xi(t) = (\xi_{ij}(t))_{1 \leq i, j \leq N}$, with $\lambda_1(t) \leq \lambda_2(t) \leq \dots \leq \lambda_N(t)$. Then let $U(t) = (u_{ij}(t))_{1 \leq i, j \leq N}$ be a unitary-matrix-valued process, which diagonalizes $\Xi(t)$,

$$U(t)^* \Xi(t) U(t) = \Lambda(t) = \text{diag}(\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t)).$$

We put

$$\Gamma_{ij, k\ell}(t) dt = \left(U(t)^* d\Xi(t) U(t) \right)_{ij} \left(U(t)^* d\Xi(t) U(t) \right)_{k\ell},$$

and the bounded variation part of $(U(t)^* d\Xi(t) U(t))_{ii}$ is written as $d\Upsilon_i(t)$. Then we introduce the Markov times

$$\begin{aligned} \sigma_{ij} &= \inf\{t \geq 0 : \lambda_i(t) \neq \lambda_j(t)\}, \\ \tau_{ij} &= \inf\{t > \sigma_{ij} : \lambda_i(t) = \lambda_j(t)\}, \quad \tau = \min_{1 \leq i < j \leq N} \tau_{ij}. \end{aligned}$$

[Generalized Bru's theorem] ([8, 9, 39, 40]) Let $\xi_{ij}(t)$, $1 \leq i, j \leq N$ be complex-valued continuous semi-martingales. Then the eigenvalue process $\boldsymbol{\lambda}(t)$ of $\Xi(t)$ solves the following system of stochastic differential equations:

$$d\lambda_i(t) = dM_i(t) + dJ_i(t), \quad t \in (0, \tau), \quad 1 \leq i \leq N.$$

where $\mathbf{M}(t) = (M_1(t), M_2(t), \dots, M_N(t))$ is the martingale with $dM_i(t)dM_j(t) = \Gamma_{ii, jj}(t)dt$, and $\mathbf{J}(t) = (J_1(t), J_2(t), \dots, J_N(t))$ is the process with bounded variation given by

$$dJ_i(t) = \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \frac{1}{\lambda_i(t) - \lambda_j(t)} \mathbf{1}_{\{\lambda_i(t) \neq \lambda_j(t)\}} \Gamma_{ij, ji}(t) dt + d\Upsilon_i(t).$$

Here $\mathbf{1}_{\{\omega\}}$ denotes an indicator function of a condition ω .

Most of the examples shown in the next subsection are systems such that all particles start from the origin and rapidly separate from each other to avoid collision. In these systems $\sigma_{ij} = 0$, $1 \leq i, j \leq N$, and $\tau = \infty$, that is, the repulsive forces among particles are strong enough to prevent any collision. For instance, in the stochastic differential equations of Dyson's Brownian motion models (3.5), the repulsive force becomes stronger as the parameter β becomes larger, and it is shown that $\tau < \infty$, if $0 < \beta < 1$, and $\tau = \infty$, if $\beta \geq 1$ [63]. This corresponds to the fact that the Bessel process is transient, if the dimension $d \geq 2$ ($\nu \geq 0$), and it is recurrent, if $0 < d < 2$ ($-1 < \nu < 0$).

4.2 Examples

In this subsection we give examples of eigenvalue processes obtained by the generalized Bru's theorem.

Let $\nu \in \mathbf{N}_0 \equiv \{0, 1, 2, \dots\}$, $B_{ij}(t)$, $\tilde{B}_{ij}(t)$, $1 \leq i \leq N + \nu$, $1 \leq j \leq N$ be independent one-dimensional Brownian motions, and $s(t)$ and $a(t)$ be $N \times N$ matrices whose element are given by

$$s_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}}B_{ij}(t), & \text{if } i < j, \\ B_{ii}(t), & \text{if } i = j, \\ \frac{1}{\sqrt{2}}B_{ji}(t), & \text{if } i > j, \end{cases} \quad a_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}}\tilde{B}_{ij}(t), & \text{if } i < j, \\ 0, & \text{if } i = j, \\ -\frac{1}{\sqrt{2}}\tilde{B}_{ji}(t), & \text{if } i > j, \end{cases}$$

respectively.

(i) GUE process

Consider the $\mathcal{H}(N)$ -valued process defined by $\Xi^{\text{GUE}}(t) = s(t) + \sqrt{-1}a(t)$, $t \in [0, \infty)$. For any fixed $t \in [0, \infty)$, $\Xi^{\text{GUE}}(t)$ is the $\mathcal{H}(N)$ -valued random variable whose probability density function with respect to the volume element $\mathcal{U}(dH)$ of $\mathcal{H}(N)$ is

$$\mu^{\text{GUE}}(H, t) = \frac{t^{-N^2/2}}{c_1(N)} \exp\left(-\frac{1}{2t}\text{Tr}H^2\right), \quad H \in \mathcal{H}(N),$$

where $\text{Tr}A$ represents the trace of a matrix A , and $c_1(N) = 2^{N/2}\pi^{N^2/2}$. We denote the group of $N \times N$ unitary matrices by $\mathbf{U}(N)$. The probability $\mu^{\text{GUE}}(H, t)\mathcal{U}(dH)$ is invariant under any unitary transformation $H \rightarrow U^*HU$ for any $U \in \mathbf{U}(N)$. In the random matrix theory, such a statistical ensemble of $\mathcal{H}(N)$ -valued random variables is called the **Gaussian unitary ensemble, GUE** [50, 51]. The probability density of eigenvalues of GUE is given by

$$g^{\text{GUE}}(\mathbf{x}, t) = \frac{t^{-N/2}}{C_1(N)} h_N\left(\frac{\mathbf{x}}{\sqrt{t}}\right)^2 \exp\left(-\frac{|\mathbf{x}|^2}{2t}\right)$$

for $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbf{W}_N^{\mathbf{A}}$ [50, 51]. Here $C_1(N)$ is the same constant as $C_1(N)$ in (3.2). By applying the generalized Bru's theorem to $\Xi^{\text{GUE}}(t)$, we see that

$\boldsymbol{\lambda}(t), t \in (0, \infty)$ solves the system of stochastic differential equations of Dyson's Brownian motion model (3.5) with $\beta = 2$. In Section 3.2 it was shown that the noncolliding Brownian motion $\mathbf{Y}(t), t \in (0, \infty)$ solves the same equation. Then the equivalence in distribution of the noncolliding Brownian motion $\mathbf{Y}(t)$ and the eigenvalue process $\boldsymbol{\lambda}(t)$ of $\Xi^{\text{GUE}}(t)$ is established.

(ii) **GOE process**

Consider the $\mathcal{S}(N)$ -valued process defined by $\Xi^{\text{GOE}}(t) = s(t)$, $t \in [0, \infty)$. For any fixed $t \in [0, \infty)$, $\Xi^{\text{GOE}}(t)$ is the $\mathcal{S}(N)$ -valued random variable whose probability density function with respect to the volume element $\mathcal{V}(dS)$ of $\mathcal{S}(N)$ is given by

$$\mu^{\text{GOE}}(S, t) = \frac{t^{-N(N+1)/4}}{c_2(N)} \exp\left(-\frac{1}{2t} \text{Tr} S^2\right), \quad S \in \mathcal{S}(N),$$

where $c_2(N) = 2^{N/2} \pi^{N(N+1)/4}$. We denote the group of $N \times N$ real symmetric matrices by $\mathbf{O}(N)$. The probability $\mu^{\text{GOE}}(S, t) \mathcal{V}(dS)$ is invariant under any orthogonal transformation $S \rightarrow {}^t V S V$ for any $V \in \mathbf{O}(N)$. Such a statistical ensemble of $\mathcal{S}(N)$ -valued random variables is called the **Gaussian orthogonal ensemble, GOE**. The probability density of eigenvalues of GOE is given by

$$g^{\text{GOE}}(\mathbf{x}, t) = \frac{t^{-N/2}}{C_2(N)} h_N\left(\frac{\mathbf{x}}{\sqrt{t}}\right) \exp\left(-\frac{|\mathbf{x}|^2}{2t}\right) \quad (4.1)$$

for $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbf{W}_N^{\mathbf{A}}$ [50, 51]. Here $C_2(N)$ is the same constant as $C_2(N)$ in (3.2). By applying the generalized Bru's theorem to $\Xi^{\text{GOE}}(t)$, we see that $\boldsymbol{\lambda}(t), t \in (0, \infty)$ solves the system of stochastic differential equations of Dyson's Brownian motion model (3.5) with $\beta = 1$.

(iii) **Laguerre process**

Let $\nu \in \mathbf{N}_0$. We denote the space of $(N + \nu) \times N$ complex matrices by $\mathcal{M}(N + \nu, N; \mathbf{C})$. Consider the $\mathcal{M}(N + \nu, N; \mathbf{C})$ -valued process defined by $L(t) = (B_{ij}(t) + \sqrt{-1} \tilde{B}_{ij}(t))_{1 \leq i \leq N+\nu, 1 \leq j \leq N}$. For any fixed $t \in [0, \infty)$, $L(t)$ is the $\mathcal{M}(N + \nu, N; \mathbf{C})$ -valued random variable whose density is given by

$$\mu_{\nu}^{\text{chGUE}}(L, t) = \frac{t^{-N(N+\nu)}}{c_3(N)} \exp\left(-\frac{1}{2t} \text{Tr} L^* L\right), \quad L \in \mathcal{M}(N + \nu, N; \mathbf{C}),$$

where $c_3(N) = (2\pi)^{N(N+\nu)}$. Such a statistical ensemble is called the **chiral Gaussian unitary ensemble, chGUE** [51]. And the $\mathcal{H}(N)$ -valued process defined by $\Xi^{\text{L}}(t) = L(t)^* L(t)$, $t \in [0, \infty)$ is called the **Laguerre process** [46]. The matrix $\Xi^{\text{L}}(t)$ is nonnegative definite and has only nonnegative eigenvalues. Applying the generalized Bru's theorem, we see that the eigenvalue process $\boldsymbol{\lambda}(t)$ solves the following system of stochastic differential equations:

$$\lambda_i(t) = 2 \int_0^t \sqrt{\lambda_i(s)} dB_i(s) + 2(N + \nu)t + 2 \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \int_0^t \frac{\lambda_i(s) + \lambda_j(s)}{\lambda_i(s) - \lambda_j(s)} ds, \quad 1 \leq i \leq N,$$

and $\tau = \infty$. Moreover, by Ito's formula we can prove that $\boldsymbol{\kappa}(t) = (\kappa_1(t), \dots, \kappa_N(t)) \equiv (\sqrt{\lambda_1(t)}, \dots, \sqrt{\lambda_N(t)})$ solves the system of stochastic differential equations (3.10), which implies the equivalence of $\boldsymbol{\kappa}(t)$ with the noncolliding $2(\nu + 1)$ -dimensional Bessel process $\mathbf{Y}^{(\nu)}(t)$. Since ν is nonnegative integer for the chGUE and the Laguerre process, the dimension $2(\nu + 1)$ of the corresponding noncolliding Bessel process is positive and even [46].

(iv) Wishart process

Let $\nu \in \mathbf{N}_0$ and denote the space of $(N + \nu) \times N$ real matrices by $\mathcal{M}(N + \nu, N; \mathbf{R})$. Consider the $\mathcal{M}(N + \nu, N; \mathbf{R})$ -valued process $W(t) = (B_{ij}(t))_{1 \leq i \leq N + \nu, 1 \leq j \leq N}$. For any fixed $t \in [0, \infty)$, $W(t)$ is the $\mathcal{M}(N + \nu, N; \mathbf{R})$ -valued random variable, whose probability density function is given by

$$\mu_\nu^{\text{chGOE}}(W, t) = \frac{t^{-N(N+\nu)/2}}{c_4(N)} \exp\left(-\frac{1}{2t} \text{Tr } {}^t W W\right), \quad W \in \mathcal{M}(N + \nu, N; \mathbf{R}),$$

where $c_4(N) = (2\pi)^{N(N+\nu)/2}$. Such a statistical ensemble is called the **chiral Gaussian orthogonal ensemble, chGOE** [51]. The $\mathcal{S}(N)$ -valued process defined by $\Xi^W(t) = {}^t W(t) W(t)$, $t \in [0, \infty)$ is called the **Wishart process** [9]. We can show that the eigenvalue process $\boldsymbol{\lambda}(t)$ of the Wishart process $\Xi^W(t)$ solves the following system of stochastic differential equations

$$\lambda_i(t) = 2 \int_0^t \sqrt{\lambda_i(s)} dB_i(s) + (N + \nu)t + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \int_0^t \frac{\lambda_i(s) + \lambda_j(s)}{\lambda_i(s) - \lambda_j(s)} ds, \quad 1 \leq i \leq N$$

and $\tau = \infty$.

We are able to apply the generalized Bru's theorem to Hermitian-matrix-valued processes with additional symmetries. We introduce the matrix σ_0 and the Pauli spin matrices $\sigma_i, i = 1, 2, 3$ by

$$\sigma_0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose that $N \geq 2$ and define $2N \times 2N$ matrices $\Sigma_\rho = I_N \otimes \sigma_\rho$, $\rho = 0, 1, 2, 3$. By definition $\Sigma_0 = I_{2N}$. Let $s^\rho(t) = (s_{ij}^\rho(t))_{1 \leq i, j \leq N}$, $a^\rho(t) = (a_{ij}^\rho(t))_{1 \leq i, j \leq N}$, $0 \leq \rho \leq 3$ be independent copies of $s(t)$, $a(t)$. By using them the $\mathcal{H}(2N)$ -process $\Xi(t)$ is divided into $2 \times 4 = 8$ terms:

$$\Xi(t) = \sum_{\rho=0}^3 \left\{ (s^\rho(t) \otimes \sigma_\rho) + (\sqrt{-1} a^\rho(t) \otimes \sigma_\rho) \right\}.$$

We consider the four Hermitian-matrix-valued processes represented by the following four terms,

$$\Xi_{\theta\varepsilon}(t) = \sum_{\rho=0}^3 (\xi_{\theta\varepsilon}^\rho(t) \otimes \sigma_\rho), \quad \theta = 1, 2, \quad \varepsilon = \pm,$$

where

$$\begin{aligned}(\xi_{\theta+}^\rho(t)) &= \begin{cases} s^\rho(t), & \text{if } \theta = 1, \rho \neq 3 \text{ or } \theta = 2, \rho = 0, \\ \sqrt{-1}a^\rho(t), & \text{if } \theta = 1, \rho = 3 \text{ or } \theta = 2, \rho \neq 0, \end{cases} \\(\xi_{\theta-}^\rho(t)) &= \begin{cases} \sqrt{-1}a^\rho(t), & \text{if } \theta = 1, \rho \neq 3 \text{ or } \theta = 2, \rho = 0, \\ s^\rho(t), & \text{if } \theta = 1, \rho = 3 \text{ or } \theta = 2, \rho \neq 0. \end{cases}\end{aligned}$$

Putting $\mathcal{H}_{\theta\varepsilon}(2N) = \{H \in \mathcal{H} : {}^t\Sigma_\theta = \varepsilon\Sigma_\theta H\}$, we see that $\Xi_{\theta\varepsilon}(t)$ takes values in $\mathcal{H}_{\theta\varepsilon}(2N)$. Due to the symmetries of matrices, eigenvalues have the following properties;

- (i) When $\varepsilon = +$, they are pairwise degenerated; $\boldsymbol{\lambda} = (\omega_1, \omega_1, \omega_2, \omega_2, \dots, \omega_N, \omega_N)$.
- (ii) When $\varepsilon = -$, they are in the form $\boldsymbol{\lambda} = (\omega_1, -\omega_1, \omega_2, -\omega_2, \dots, \omega_N, -\omega_N)$.

(v) GSE process

In the case of $(\theta, \varepsilon) = (2, +)$, a matrix $\Xi \in \mathcal{H}_{2+}(2N)$ is said to be a **self-dual** Hermitian matrix, if it has the symmetry ${}^t\Xi\Sigma_2 = \Sigma_2\Xi$ in addition to Hermitian property. The matrix can be diagonalized by a unitary-symplectic matrix. For any fixed $t \in (0, \infty)$ the statistical ensemble of $\Xi_{2+}(t)$ is invariant under any unitary-symplectic transformation, and is called **Gaussian symplectic ensemble, GSE**. The probability density function of eigenvalues of GSE is given by [50, 51],

$$g^{\text{GSE}}(\mathbf{x}; t) = \frac{t^{-N/2}}{C_3(N)} h_N \left(\frac{\mathbf{x}}{\sqrt{t}} \right)^4 \exp \left(-\frac{|\mathbf{x}|^2}{2t} \right), \quad \mathbf{x} \in \mathbf{W}_N^A,$$

where $C_3(N) = (2\pi)^{N/2} \prod_{i=1}^N \Gamma(2i)$. The eigenvalues are pairwise degenerated and represented as $\boldsymbol{\lambda} = (\omega_1, \omega_1, \omega_2, \omega_2, \dots, \omega_N, \omega_N)$. This is known as the Kramers doublets in quantum mechanics. Applying the generalized Bru's theorem, we see that the distinct eigenvalues $\omega_i, 1 \leq i \leq N$ solves the system of equations of Dyson's Brownian motion model (3.5) with $\beta = 4$,

$$\omega_i(t) = B_i(t) + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \int_0^t \frac{2}{\omega_i(s) - \omega_j(s)} ds, \quad 1 \leq i \leq N.$$

For a pair of degenerated eigenvalues $\lambda_{2i-1} = \lambda_{2i} = \omega_i$, $\sigma_{2i-1, 2i} = \infty$, $1 \leq i \leq N$. All other pairs separately move and never coincide with each other, that is, $\tau = \infty$.

(vi) Matrix-valued process of class C

In the case of $(\theta, \varepsilon) = (2, -)$, a matrix $\Xi \in \mathcal{H}_{2-}(2N)$ has the symmetry ${}^t\Xi(t)\Sigma_2 = -\Sigma_2\Xi(t)$ in addition to Hermitian property. We denote by $\text{sp}(2N; \mathbf{C})$ the Lie algebra of complex symplectic group represented by $2N \times 2N$ matrices. Then $\mathcal{H}_{2-}(2N) \simeq \text{sp}(2N; \mathbf{C}) \cap \mathcal{H}(2N)$. For fixed $t \in (0, \infty)$ the statistical ensemble of $\Xi_{2-}(t)$ coincides with the random matrix ensemble called **class C** introduced by Altland and Zirnbauer [2]. The eigenvalues of a matrix in this class are in the form

$\lambda = (\omega_1, -\omega_1, \omega_2, -\omega_2, \dots, \omega_N, -\omega_N)$. We denote the increasing sequence of non-negative eigenvalues by $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_N(t))$. By the generalized Bru's theorem, we see that $\omega(t)$ solves the following system of equations

$$\omega_i(t) = B_i(t) + \int_0^t \frac{1}{\omega_i(s)} ds + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \int_0^t \left\{ \frac{1}{\omega_i(s) - \omega_j(s)} + \frac{1}{\omega_i(s) + \omega_j(s)} \right\} ds, \\ 1 \leq i \leq N.$$

It is also verified that $\tau = \infty$ and $\omega(t) \in \mathbf{W}_N^{\mathbf{C}}, \forall t \in (0, \infty)$ with probability one. Comparing the above system of equations with (3.10), we can conclude the equivalence of $\omega(t)$ and the noncolliding three-dimensional Bessel process $\mathbf{Y}^{(1/2)}(t)$, in distribution [40]. Moreover, it coincides with the noncolliding Brownian motion under the condition that it never hits the wall at the origin [40].

(vii) Matrix-valued process of class D

In the case of $(\theta, \varepsilon) = (1, -)$, a matrix $\Xi \in \mathcal{H}_{1-}(2N)$ has the symmetry ${}^t\Xi(t)\Sigma_1 = -\Sigma_1\Xi(t)$ in addition to Hermitian property. We denote by $\mathfrak{so}(2N; \mathbf{C})$ the complexification of Lie algebra of special orthogonal group represented by $2N \times 2N$ matrices. Then $\mathcal{H}_{1-}(2N) \simeq \mathfrak{so}(2N; \mathbf{C}) \cap \mathcal{H}(2N)$. For any fixed $t \in (0, \infty)$ the statistical ensemble of $\Xi_{1-}(t)$ coincides with the random matrix ensemble called **class D** introduced by Altland and Zirnbauer [2]. The eigenvalues of a matrix in this class are also in the form $\lambda = (\omega_1, -\omega_1, \omega_2, -\omega_2, \dots, \omega_N, -\omega_N)$. We denote the increasing sequence of nonnegative eigenvalues by $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_N(t))$. By the generalized Bru's theorem, we see that $\omega(t)$ solves the following system of equations

$$\omega_i(t) = B_i(t) + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \int_0^t \left\{ \frac{1}{\omega_i(s) - \omega_j(s)} + \frac{1}{\omega_i(s) + \omega_j(s)} \right\} ds, \quad 1 \leq i \leq N.$$

It is also verified that $\tau = \infty$ and $\omega(t) \in \mathbf{W}_N^{\mathbf{D}}, \forall t \in (0, \infty)$ with probability one. Comparing the above system of equations with (3.10), we can conclude that $\omega(t)$ is equivalent in distribution with the noncolliding one-dimensional Bessel process $\mathbf{Y}^{(-1/2)}(t)$ [40]. Since one-dimensional Bessel process is identified with a reflecting Brownian motion, $\omega(t)$ can be also regarded as the noncolliding reflecting Brownian motion [40].

5 Determinantal processes

5.1 Fredholm determinant

Let \mathcal{X} be the space of countable subsets of \mathbf{R} without accumulation points. For $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N$ we write $\{\mathbf{x}\}$ for an element $\{x_1, x_2, \dots, x_N\}$ of \mathcal{X} . For $\mathbf{x}_N \in \mathbf{R}^N$ and $N' \in \{1, 2, \dots, N\}$, we write $\mathbf{x}_{N'}$ for $(x_1, x_2, \dots, x_{N'}) \in \mathbf{R}^{N'}$. For the

temporally homogeneous noncolliding Brownian motion $\mathbf{Y}(t)$, the \mathcal{X} -valued process $\xi^N(t) = \{\mathbf{Y}(t)\}$ has the transition probability density function

$$\tilde{p}_N(s, \{\mathbf{x}\}; t, \{\mathbf{y}\}) = \begin{cases} p_N(t-s, \mathbf{y}|\mathbf{x}), & \text{if } s > 0, \mathbf{x}, \mathbf{y} \in \mathbf{W}_N^A, \\ p_N(t, \mathbf{y}|\mathbf{0}), & \text{if } s = 0, \mathbf{x} = \mathbf{0}, \mathbf{y} \in \mathbf{W}_N^A, \\ 0, & \text{otherwise.} \end{cases}$$

We call the process $\xi^N(t)$ also the temporally homogeneous noncolliding Brownian motion in this paper. The \mathcal{X} -valued noncolliding Bessel process $\xi^{N,\nu}(t)$ can be defined as well. For a sequence of time $0 < t_1 < \dots < t_M = T$ and a sequence of positive integers less than or equal to N , $\{N_m\}_{m=1}^M$, the **multi-time correlation function** of $\xi^N(\cdot)$ at $(t_m, \{\mathbf{x}_{N_m}^{(m)}\})$, $m = 1, 2, \dots, M$, is given by

$$\begin{aligned} & \rho_N \left(t_1, \mathbf{x}_{N_1}^{(1)}; t_2, \mathbf{x}_{N_2}^{(2)}; \dots; t_M, \mathbf{x}_{N_M}^{(M)} \right) \\ &= \int_{\prod_{m=1}^M \mathbf{R}^{N-N_m}} \prod_{m=1}^M \frac{1}{(N-N_m)!} \prod_{i=N_m+1}^N dx_i^{(m)} \prod_{\ell=0}^{M-1} \tilde{p}_N(t_\ell, \{\mathbf{x}_N^{(\ell)}\}; t_{\ell+1}, \{\mathbf{x}_N^{(\ell+1)}\}), \end{aligned}$$

where we put $t_0 = 0$, $\mathbf{x}_N^{(0)} = \mathbf{0}$. Let $C_0(\mathbf{R})$ be the set of all real continuous functions with compact supports. For $\mathbf{f} = (f_1, f_2, \dots, f_M) \in C_0(\mathbf{R})^M$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_M) \in \mathbf{R}^M$, we put $\chi_m(x) = e^{\theta_m f_m(x)} - 1$, $1 \leq m \leq M$ and $\boldsymbol{\chi} = (\chi_1, \chi_2, \dots, \chi_M)$. The **multi-time moment generating function**

$$\Psi_N(\boldsymbol{\chi}; \boldsymbol{\theta}) \equiv E \left[\exp \left\{ \sum_{m=1}^M \theta_m \sum_{i_m=1}^N f_m(X_{i_m}(t_m)) \right\} \right]$$

of $\xi^N(t)$, $t \in [0, T]$ can be expanded by means of the multi-time correlation functions as follows:

$$\begin{aligned} & \sum_{N_1=0}^N \sum_{N_2=0}^N \dots \sum_{N_M=0}^N \prod_{m=1}^M \frac{1}{N_m!} \int_{\mathbf{R}^{N_1}} \prod_{i=1}^{N_1} dx_i^{(1)} \int_{\mathbf{R}^{N_2}} \prod_{i=1}^{N_2} dx_i^{(2)} \dots \int_{\mathbf{R}^{N_M}} \prod_{i=1}^{N_M} dx_i^{(M)} \\ & \times \prod_{m=1}^M \prod_{i=1}^{N_m} \chi_m(x_i^{(m)}) \rho_N \left(t_1, \mathbf{x}_{N_1}^{(1)}; t_2, \mathbf{x}_{N_2}^{(2)}; \dots; t_M, \mathbf{x}_{N_M}^{(M)} \right). \end{aligned}$$

An \mathcal{X} -valued process $\xi(t)$ is called a **determinantal process**, if its multi-time moment generating function is written by **Fredholm determinant** as

$$\Psi(\boldsymbol{\chi}; \boldsymbol{\theta}) = \text{Det} \left[\delta_{m,n} \delta(x-y) + \mathbf{K}(t_m, x; t_n, y) \chi_n(y) \right] \quad (5.1)$$

with a locally integrable function \mathbf{K} . We call the function \mathbf{K} the correlation kernel of the process. By definition of Fredholm determinant, the multi-time correlation function of $\xi(t)$ is then given by

$$\rho_N \left(t_1, \mathbf{x}_{N_1}^{(1)}; t_2, \mathbf{x}_{N_2}^{(2)}; \dots; t_M, \mathbf{x}_{N_M}^{(M)} \right) = \det \left[\mathbf{A} \left(\mathbf{x}_{N_0}^{(0)}, \mathbf{x}_{N_1}^{(1)}, \dots, \mathbf{x}_{N_M}^{(M)} \right) \right]$$

diffusions	Wyle chambers	matrix-valued pr.	RM	orth. poly.
Brownian motion	A_{N-1}	GUE $\mathcal{H}(N)$	GUE	H_n
even-dim. Bessel pr.	C_N	Laguerre	chGUE	$L_n^\nu, \nu \in \mathbf{N}_0$
3-dim. Bessel pr. (absorbing BM)	C_N	class C $\text{sp}(2N; \mathbf{C}) \cap \mathcal{H}(2N)$	class C	$L_n^{1/2}$
1-dim. Bessel pr. (reflecting BM)	D_N	class D $\text{so}(2N; \mathbf{C}) \cap \mathcal{H}(2N)$	class D	$L_n^{-1/2}$

Table 1: Noncolliding diffusion processes, random matrix ensembles, and orthogonal polynomials

with a $\sum_{m=1}^M N_m \times \sum_{m=1}^M N_m$ matrix

$$\mathbf{A}(\mathbf{x}_{N_1}^{(1)}, \mathbf{x}_{N_2}^{(2)}, \dots, \mathbf{x}_{N_M}^{(M)}) = \left(\mathbf{K}(t_m, x_i^{(m)}; t_n, x_j^{(n)}) \right)_{1 \leq i \leq N_m, 1 \leq j \leq N_n, 1 \leq m, n \leq M}.$$

The noncolliding Brownian motion $\xi^N(t)$ is the determinantal process with the correlation kernel \mathbf{K}_N :

$$\mathbf{K}_N(s, x; t, y) = \begin{cases} \frac{1}{\sqrt{2s}} \sum_{n=0}^{N-1} \left(\frac{t}{s}\right)^{n/2} \varphi_n\left(\frac{x}{\sqrt{2s}}\right) \varphi_n\left(\frac{y}{\sqrt{2t}}\right), & \text{if } s \leq t, \\ -\frac{1}{\sqrt{2s}} \sum_{n=N}^{\infty} \left(\frac{t}{s}\right)^{n/2} \varphi_n\left(\frac{x}{\sqrt{2s}}\right) \varphi_n\left(\frac{y}{\sqrt{2t}}\right), & \text{if } s > t, \end{cases}$$

where $\varphi_n(x) = \{\sqrt{\pi}2^n n!\}^{-1/2} H_n(x) e^{-x^2/2}$, $n = 0, 1, 2, \dots$, are the orthonormal functions on \mathbf{R} associated with the Hermite polynomials $H_n(x)$ [37, 43]. The noncolliding Bessel process $\xi^{N,\nu}(t)$ is the determinantal process with the correlation kernel $\mathbf{K}_N^{(\nu)}$:

$$\mathbf{K}_N^{(\nu)}(s, x; t, y) = \begin{cases} \frac{\sqrt{xy}}{s} \sum_{n=0}^{N-1} \left(\frac{t}{s}\right)^n \varphi_n^\nu\left(\frac{x^2}{2s}\right) \varphi_n^\nu\left(\frac{y^2}{2t}\right), & \text{if } s \leq t, \\ -\frac{\sqrt{xy}}{s} \sum_{n=N}^{\infty} \left(\frac{t}{s}\right)^n \varphi_n^\nu\left(\frac{x^2}{2s}\right) \varphi_n^\nu\left(\frac{y^2}{2t}\right), & \text{if } s > t, \end{cases}$$

where $\varphi_n^\nu(x) = \sqrt{\Gamma(n+1)/\Gamma(\nu+n+1)} x^{\nu/2} L_n^\nu(x) e^{-x/2}$, $n = 0, 1, 2, \dots$, are the orthonormal functions on $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$ associated with the Laguerre polynomials $L_n^\nu(x)$ with parameter $\nu > -1$ [42]. See Table 1, which summarizes the correspondence between noncolliding diffusion processes with finite number of particles, statistical ensembles of random matrices (RM), and orthogonal polynomials used to represent correlation kernels.

5.2 Scaling limits

When the number of the diffusion process N goes to infinity, the asymptotic behaviors of the noncolliding processes ξ^N and $\xi^{N,\nu}$ are determined by the asymptotics of their correlation kernels $\mathbf{K}_N, \mathbf{K}_N^{(\nu)}$ in $N \rightarrow \infty$. Suppose that the correlation kernel converges under an appropriate scaling limit. Then the multi-time moment generating function $\Psi_N(\boldsymbol{\chi}; \boldsymbol{\theta})$ and the multi-time correlation functions ρ_N converge, and then the process converges in the sense of finite dimensional distributions. In the following, we discuss the **bulk scaling limit** and the **soft-edge scaling limit** for the noncolliding Brownian motion $\mathbf{Y}(t)$ and the **hard-edge scaling limit** for the noncolliding Bessel process $\mathbf{Y}^{(\nu)}(t)$ [52, 60, 34, 72, 37, 1, 42, 43].

1. **[Bulk scaling limit]** As $N \rightarrow \infty$, the process $\xi^N(N+t)$ converges to the infinite-dimensional determinantal process, whose correlation kernel \mathcal{K}^{sin} is expressed by using trigonometric functions,

$$\mathcal{K}^{\text{sin}}(s, x; t, y) = \begin{cases} \frac{1}{\pi} \int_0^1 du e^{(t-s)u^2/2} \cos(u(x-y)), & \text{if } s < t, \\ \frac{\sin(x-y)}{\pi(x-y)}, & \text{if } s = t, \\ -\frac{1}{\pi} \int_1^\infty du e^{(t-s)u^2/2} \cos(u(x-y)), & \text{if } s > t. \end{cases}$$

2. **[Soft-edge scaling limit]** Define the scaled process $\theta_{a(N,t)} \xi^N(N^{1/3} + t) \equiv \{Y_1(N^{1/3} + t) - a(N, t), Y_2(N^{1/3} + t) - a(N, t), \dots, Y_N(N^{1/3} + t) - a(N, t)\}$ with $a(N, t) = 2N^{2/3} + N^{1/3}t - t^2/4$. As $N \rightarrow \infty$, it converges to the infinite-dimensional determinantal process, whose correlation kernel \mathcal{K}^{Ai} is expressed by using the Airy function $\text{Ai}(x)$,

$$\mathcal{K}^{\text{Ai}}(s, x; t, y) = \begin{cases} \int_{-\infty}^0 du e^{(t-s)u/2} \text{Ai}(x-u) \text{Ai}(y-u), & \text{if } s \leq t, \\ -\int_0^\infty du e^{(t-s)u/2} \text{Ai}(x-u) \text{Ai}(y-u), & \text{if } s > t. \end{cases}$$

3. **[Hard-edge scaling limit]** As $N \rightarrow \infty$, $\xi^{N,\nu}(N+t)$ converges to the infinite-dimensional determinantal process, whose correlation kernel $\mathcal{K}^{(\nu)}$ is

expressed by using the Bessel function $J_\nu(x)$,

$$\mathcal{K}^{(\nu)}(s, x; t, y) = \begin{cases} \sqrt{xy} \int_0^2 du e^{(t-s)u^2/2} J_\nu(ux) u J_\nu(uy), & \text{if } s < t, \\ \frac{2\sqrt{xy} \{J_\nu(2x)yJ'_\nu(2y) - J_\nu(2y)xJ'_\nu(2x)\}}{x^2 - y^2}, & \text{if } s = t, \\ -\sqrt{xy} \int_2^\infty du e^{(t-s)u^2/2} J_\nu(ux) u J_\nu(uy), & \text{if } s > t. \end{cases}$$

The above three infinite particle systems are all temporally homogeneous. The system obtained by the bulk scaling limit is spatially homogeneous, and the other systems obtained by the soft- and hard-edge scaling limits are spatially inhomogeneous (see Table 2). These infinite particle systems are reversible and their equilibrium measures are determinantal point processes [43, 45]. Osada [56, 57] constructed diffusion processes whose equilibrium measures are determinantal point processes, by the Dirichlet form technique. Although the coincidence of Osada's processes and the above processes is expected [43], it has not been proved yet. If the coincidence were proved, it would be concluded that the infinite particle systems obtained by the bulk scaling limit and the hard-edge scaling limit solve the stochastic differential equations (3.5) and (3.10) with $N = \infty$ (see [58]). Nonequilibrium dynamics of determinantal processes with infinite numbers of particles have been studied, which show the relaxation processes to the stationary determinantal processes with the correlation kernels \mathcal{K}^{sin} and \mathcal{K}^{Ai} [44, 45]. There the theory of distributions of zeros and orders of growth of entire functions [47] is applied to analyze the determinantal structures of noncolliding diffusion processes with infinite numbers of particles.

5.3 Tracy-Widom distribution

Consider the motion of the rightmost particle in the temporally homogeneous non-colliding Brownian motion $\xi^N(t) = \{\mathbf{Y}(t)\}$. For a fixed time $t > 0$, from (5.1) with $M = 1, t_1 = t, \theta_1 = 1, \chi_1(x) = -\mathbf{1}_{\{x > \alpha\}}$ the probability that the position of the rightmost particle is less than $\alpha \in \mathbf{R}$ is given by the Fredholm determinant as

$$\begin{aligned} P\left(\max_{1 \leq i \leq N} Y_i(t) \leq \alpha\right) &= E\left[\exp\left\{\sum_{i=1}^N \log(\mathbf{1}_{\{Y_i(t) \leq \alpha\}})\right\}\right] \\ &= \text{Det}\left[\delta(x - y) - \mathbf{K}_N(t, x; t, y) \mathbf{1}_{\{y > \alpha\}}\right]. \end{aligned}$$

Then the distribution function $F_{\text{max}}(\alpha)$ of the position of the rightmost particle in the process obtained by the soft-edge scaling limit is given by

$$F_{\text{max}}(\alpha) \equiv \text{Det}\left[\delta(x - y) - \mathcal{K}^{\text{Ai}}(t, x; t, y) \mathbf{1}_{\{y > \alpha\}}\right],$$

kernels (finite system)	Hermite H_n	Hermite H_n	Laguerre L_n^ν
kernels (infinite system)	(bulk) trigonometric sin, cos	(soft-edge) Airy Ai	(hard-edge) Bessel J_ν
spatial homogeneity	homogeneous	inhomogeneous	inhomogeneous
Painlevé equation	P_V	P_{II}	P_{III}

Table 2: Determinantal processes and Painlevé equations

where the kernel \mathcal{K}^{Ai} was defined in the previous subsection. Tracy and Widom [70] represented the distribution function (**Tracy-Widom distribution**) as

$$F_{\max}(\alpha) = \exp \left(- \int_{\alpha}^{\infty} (x - \alpha) q(x)^2 dx \right)$$

with the solution $q(x)$ of **Painlevé II** (see for instance [54])

$$P_{II} \quad : \quad \frac{d^2 q(x)}{dx^2} = 2q(x)^3 + xq(x)$$

satisfying the boundary condition $q(x) \sim \text{Ai}(x)$, $x \rightarrow \infty$. For determinantal process, the distribution of position of the right-nearest particle to the fixed point (for instance the origin) is described by Fredholm determinant as well as the rightmost particle. In the bulk scaling limit [31] and the hard-edge scaling limit [71] the distributions of right-nearest particles to fixed points are studied precisely, and are represented by solutions of Painlevé V (P_V) and Painlevé III (P_{III}), respectively (see Table 2).

6 Temporally inhomogeneous processes

The noncolliding processes discussed in the previous section are temporally homogeneous diffusion processes, in which noncolliding conditions are imposed in the infinite time-intervals $(0, \infty)$. On the other hand, in Section 3 we explained that the noncolliding Brownian motion $\mathbf{X}(t), t \in [0, T]$ is a temporally inhomogeneous diffusion process with transition probability density function (3.1) with (3.3), if noncolliding conditions are imposed during a finite time-interval $(0, T]$. Since $\mathcal{N}_N(0, \mathbf{y}) = 1$, $\mathbf{y} \in \mathbf{W}_N^{\text{A}}$ by definition, the probability density function (3.3) of the process $\mathbf{X}(T)$ coincides with g^{GOE} given by (4.1). In other words, the distribution of the noncolliding Brownian motion at the terminal time T of the noncolliding time-interval is equal to the eigenvalue distribution of GOE. When $0 < t < T$, however, the distribution of $\mathbf{X}(t)$ is different from the eigenvalue distribution of GOE. In particular, when $0 < t \ll T$, we see that it is close to the eigenvalue distribution of GUE by the

asymptotic behavior (3.2) of $\mathcal{N}_N(t, \mathbf{y})$. From the above observations it is expected that $\mathbf{X}(t), t \in [0, T]$ exhibits a **transition from GUE to GOE** as t approaches T .

Remind that the off-diagonal elements of the GOE process $\Xi^{\text{GOE}}(t)$ are one-dimensional Brownian motions and those of GUE process $\Xi^{\text{GUE}}(t)$ are complex Brownian motions. Hence, an $\mathcal{H}(N)$ -valued process, whose eigenvalue process realizes $\mathbf{X}(t), t \in (0, T]$, should satisfy the condition that each off-diagonal element behaves like a complex Brownian motion for $0 < t \ll T$, and it becomes to behave like a real Brownian motion as $t \nearrow T$ [59]. We find that, if each imaginary part of off-diagonal element is given by the Brownian bridge of duration T , which was introduced in Section 2, this condition is fulfilled. Let $\beta_{ij}^T(t), 1 \leq i < j \leq N$ be independent Brownian bridges, which are assumed to be independent of the Brownian motions $B_{ij}(t), 1 \leq i \leq j \leq N$ used in the definition of $s_{ij}(t)$. Then we put

$$a_{ij}^T(t) = \begin{cases} \frac{1}{\sqrt{2}}\beta_{ij}^T(t), & \text{if } i < j \\ 0, & \text{if } i = j \\ -\frac{1}{\sqrt{2}}\beta_{ji}^T(t), & \text{if } i > j \end{cases}$$

and define an $\mathcal{H}(N)$ -valued process by

$$\Xi^T(t) = \left(s_{ij}(t) + \sqrt{-1}a_{ij}^T(t) \right)_{1 \leq i, j \leq N}, \quad t \in [0, T]. \quad (6.1)$$

Then we see that the eigenvalue process $\boldsymbol{\lambda}^T(t)$ of $\Xi^T(t), t \in (0, T]$ is a temporally inhomogeneous diffusion process and is equivalent in distribution with the temporally inhomogeneous noncolliding Brownian motion $\mathbf{X}(t), t \in (0, T]$ with $\mathbf{X}(0) = \mathbf{0}$ [39]. The equivalence is proved by the fact that the eigenvalue process $\boldsymbol{\lambda}^T(t)$ of $\Xi^T(t)$ and that of $\Xi^{\text{GUE}}(t)$ satisfy the generalized Imhof's relation (3.6). Remember that the eigenvalue process of $\Xi^{\text{GUE}}(t)$ is identified with the temporally homogeneous noncolliding Brownian motion $\mathbf{Y}(t)$ in distribution as shown in Section 4.

This result implies that the process $\mathbf{X}(t), t \in (0, T]$ has two different representations, ‘the representation by a noncolliding Brownian motion with the transition probability density (3.3) given by the Karlin-McGregor formula’, and ‘the representation by an eigenvalue process of $\Xi^T(t)$ given by (6.1)’. This claim is a generalization of the result that the three-dimensional Bessel process as well as the generalized meander have two different representations, ‘the representation by conditional one-dimensional Brownian motions’ and ‘the representation by radial parts of three-dimensional diffusion processes’.

The $\mathcal{H}(N)$ -valued process $\Xi^T(t)$ is decomposed into an eigenvalue part $\Lambda(t)$ and a unitary matrix part $U(t)$. The latter representation of the process $\mathbf{X}^T(t), t \in (0, T]$ implies that it is obtained from $\Xi^T(t)$ by integrating its unitary matrix part $U(t)$. By this observation the following identity is derived [39], which is called the **Harish-Chandra integral formula** [24] or the **Itzykson-Zuber integral formula** [29].

[Harish-Chandra integral formula] Let dU be the Haar measure of the space $\mathbf{U}(N)$ normalized as $\int_{\mathbf{U}(N)} dU = 1$. For $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbf{W}_N^A$ and $\mathbf{y} = (y_1, y_2, \dots, y_N) \in \mathbf{W}_N^A$, put $\Lambda_{\mathbf{x}} = \text{diag}(x_1, x_2, \dots, x_N)$ and $\Lambda_{\mathbf{y}} = \text{diag}(y_1, y_2, \dots, y_N)$. Then for any $\sigma \in \mathbf{R}$ the following identity holds:

$$\int_{\mathbf{U}(N)} dU e^{-\text{Tr}(\Lambda_{\mathbf{x}} - U^* \Lambda_{\mathbf{y}} U)^2 / (2\sigma^2)} = \frac{C_1(N) \sigma^{N^2}}{h_N(\mathbf{x}) h_N(\mathbf{y})} \det_{1 \leq i, j \leq N} \left(G(\sigma^2, y_j | x_i) \right).$$

The above argument is also valid for the noncolliding generalized meander $\mathbf{X}^{(\nu, \kappa)}(t)$. The matrix-valued process, whose elements are the complex-valued process having Brownian motions as its real part and Brownian bridges as its imaginary part,

$$M_T(t) = \left(B_{ij}(t) + \sqrt{-1} \beta_{ij}^T(t) \right)_{1 \leq i \leq N+\nu, 1 \leq j \leq N},$$

exhibits a **transition from chGUE to chGOE**. Its eigenvalue process is a complex-valued process and different from the noncolliding generalized meander. In stead of the process $M_T(t)$, we consider the $\mathcal{H}(N)$ -valued process defined by

$$\Xi_T^{\text{LW}}(t) = M_T(t)^* M_T(t), \quad t \in [0, T]$$

The eigenvalue process $\boldsymbol{\lambda}^{\text{LW}}(t) = (\lambda_1^{\text{LW}}(t), \lambda_2^{\text{LW}}(t), \dots, \lambda_N^{\text{LW}}(t))$ of $\Xi_T^{\text{LW}}(t)$ is the stochastic process with N nonnegative coordinates. We put $\kappa_i^{\text{LW}}(t) = \sqrt{\lambda_i^{\text{LW}}(t)}$, $1 \leq i \leq N$ and consider $\boldsymbol{\kappa}^{\text{LW}}(t) = (\kappa_1^{\text{LW}}(t), \kappa_2^{\text{LW}}(t), \dots, \kappa_N^{\text{LW}}(t))$. Then the process $\boldsymbol{\kappa}^{\text{LW}}(t)$ is temporally inhomogeneous and is identified with the noncolliding generalized meander $\mathbf{X}^{(\nu, \kappa)}(t)$, $t \in (0, T]$ with $\nu \in \mathbf{N}_0$, $\kappa = \nu + 1$ and $\mathbf{X}^{(\nu, \kappa)}(0) = \mathbf{0}$ [40]. As another example, by setting $(\nu, \kappa) = (1/2, 1)$, we can construct an $\mathcal{H}(N)$ -valued process, whose eigenvalue process is a noncolliding generalized meander exhibiting a **transition from class C to class CI**. (For the definitions of the random matrix ensembles called class CI and class DIII-odd/even mentioned below, see [75, 2, 30, 10].)

For a system of $2N$ independent Brownian motions, we impose the condition that pairs of $(2i - 1)$ -th and $2i$ -th particles meet at the terminal time T for $1 \leq i \leq N$, in addition to the noncolliding condition in the time-interval $(0, T)$. Then we can show that the system realizes the eigenvalue process of the matrix-valued process, which exhibits a **transition from GUE to GSE**.

Consider the noncolliding generalized meanders with $(\nu, \kappa) = (\nu, \nu + 1)$, $\nu \in \mathbf{N}_0$, $(\nu, \kappa) = (1/2, 0)$ and $(\nu, \kappa) = (-1/2, 0)$, with the above mentioned additional condition at $t = T$. We can prove that they realize the eigenvalue processes of the matrix-valued processes, which shows transitions **from chGUE to chGSE**, **from D to class DIII-odd** and **from class D to class DIII-even**, respectively [40]. (Here chGSE indicates the random matrix ensemble called the **chiral Gaussian symplectic ensemble** [51].)

Several temporally inhomogeneous processes have two different representations, ‘the representation by noncolliding diffusion processes’, and ‘the representation by eigenvalue processes of matrix-value processes’ (see Table 3). By identifying these

homogeneity	homogeneous	inhomogeneous
1 dim. diffusion	Brownian motion Bessel process	Brownian bridge Bessel bridge generalized meander
matrix-valued pr.	GUE chGUE class C class D	GUE-to-GOE, GUE-to-GSE chGUE-to-chGOE, chGUE-to-chGSE class C-to-class CI class D-to-class DIII-odd class D-to-class DIII-even
process	determinantal pr.	Pfaffian process
corr. func.	determinant	Pfaffian
moment gen. func.	Fredholm det.	Fredholm Pfaffian

Table 3: Temporally homogeneous and inhomogeneous processes

two representations, Harish-Chandra (Itzykson-Zuber) formulas are derived for matrices with a variety of symmetries [40].

Recently, a family of stochastic processes, whose multi-time moment generating functions are represented by Fredholm Pfaffians [61, 42], has been intensively studied [67, 7]. We call such a stochastic process a **Pfaffian process**. In general, the multi-time N point correlation function of a Pfaffian process is described by a Pfaffian of $2N \times 2N$ matrix (see Table 3). Since Pfaffians of $2N \times 2N$ matrices are reduced to determinants of $N \times N$ matrices in special cases, Pfaffian process is regarded as a generalization of determinantal process. Dyson introduced stochastic processes, whose N point correlation functions are represented by $N \times N$ quaternion determinants, and they have been studied since then [15, 18, 51]. These processes are also members of Pfaffian processes, because quaternion determinants can be expressed by Pfaffians. We showed that the temporally inhomogeneous version of noncolliding Brownian motion $\mathbf{X}(t)$ and the noncolliding generalized meanders $\mathbf{X}^{(\nu, \kappa)}$ are Pfaffian processes [37, 42]. By evaluating the asymptotics of Pfaffians in $N \rightarrow \infty$, we can prove the existence of infinite-dimensional Pfaffian processes in appropriate scaling limits. They describe temporally inhomogeneous infinite particle systems. For the noncolliding generalized meander $\mathbf{X}^{(\nu, \kappa)}$, the general form of correlation kernel is described by using **Riemann-Liouville differintegrals** [42].

7 Miscellanea

1. In this paper, we have discussed noncolliding systems of one-dimensional diffusion processes in the unbounded domains, \mathbf{R} and \mathbf{R}_+ , which are related to Gaussian ensembles of random matrices. We can also consider noncolliding systems in

bounded domains. In particular, the systems on a circle have been studied and the relation with the statistical ensembles of random unitary matrices called **circular ensembles** are reported [13, 25, 50, 53]. For the systems on finite intervals, the transition probability density functions are described by using the Jacobi polynomials and the systems are related to the random matrix model called **MANOVA (multivariate analysis of variance) model** [12].

2. Dyson's Brownian motion models, which solve the system of equations (3.5), form a family of processes with a parameter $\beta > 0$. In this paper by applying the generalized Bru's theorem we have clarified the correspondence between the eigenvalue processes associated with the random matrix ensembles GOE, GUE and GSE, and the Dyson's Brownian motion models with $\beta = 1, 2$ and 4. In particular, it was shown that, when $\beta = 2$, the process is also realized by the noncolliding Brownian motion. Recently, a family of random matrix ensemble with a parameter $\beta > 0$ is proposed, in which the eigenvalue distribution is give by

$$g^\beta(\mathbf{x}) = \frac{1}{C_\beta(N)} h_N(\mathbf{x})^\beta \exp\left(-\frac{|\mathbf{x}|^2}{2}\right), \quad \mathbf{x} \in \mathbf{W}_N^A,$$

where $C_\beta(N)$ is the normalization constant. This random matrix ensemble is called the **Gaussian beta ensemble**, whose elements are tridiagonal matrices such that diagonal elements are independent Gaussian random variables and $(k, k+1)$ -elements and $(k+1, k)$ -elements, $1 \leq k \leq N-1$, are independent random variables with χ -square distribution with degree-of-freedom $(N-k)\beta$ [12].

3. The ensembles of random matrices, whose elements are independent complex Gaussian random variables, is called the **Ginibre ensemble** [22]. The eigenvalues of matrices in this ensemble are complex in general and the probability density function is given by

$$g^{\text{Gin}}(\mathbf{z}) = \frac{1}{C_{\text{Gin}}(N)} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 \exp\left(-\frac{|\mathbf{z}|^2}{2}\right), \quad \mathbf{z} \in \mathbf{C}^N,$$

where $C_{\text{Gin}}(N)$ is the normalization constant. Characterization of $g^{\text{Gin}}(\mathbf{z})$ has been intensively studied (see for instance [64, 58]).

4. In the present paper noncolliding diffusion processes are discussed. Noncolliding systems of discrete time Markov processes have been also studied. In particular, the system of noncolliding random walks, called the **vicious walk model** [17], is an interesting and important model, since it is related to the representation theory of symmetry groups through the Young diagrams, the Young tableaux, and the Schur functions [32, 3, 33, 38, 55, 36, 26, 5, 6].

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